https://www.desmos.com/calculator/fmcofkyssl

Predicting Projectile Motion in a Gas of Uniform Density

Wilson Malone

Imagine a projectile that is a round ball made of lead. Imagine that projectile is launched through a uniform gaseous medium at 1000 meters per second with no gravity. How does the velocity of the projectile change over time? This question has been on my mind for years, but I've recently been able to come up with a function that seems accurate to describing the velocity and position of such a projectile.

I'll start with the basic drag equation, $F_d = C_d \frac{\rho}{2} A v^2$ which describes the force acting on an object as a function of its velocity. C_d is the drag coefficient, ρ is the density of the medium, A is the cross-sectional area of the object, and v is the velocity. By dividing this function by the mass of your object, you can get an equation for drag acceleration (or deceleration) as a function of velocity. $a(v) = \frac{C_d \frac{\rho}{2} A v^2}{m}$.

Here I'll start with an iterative approach to finding the velocity over time by multiplying this acceleration function by a small unit of time Δt. In this example, I'm using .001 seconds. $\mathrm{v_f} = \mathrm{v_i} - \frac{C_d \frac{\rho}{2} A v^2}{m} \Delta t$. The attached Excel file shows a graph of the velocity over time by calculating each velocity change for 1000 iterations.

The shape of the resulting graph suggests a function with the form $v(t) = \frac{c}{t+a}$ where c and a are representing unknown constants. By trial and error (A LOT of fiddling around in Desmos) using the constants of drag coefficient, air density, area, and velocity, a function can be deduced that closely matches the values given by the iterative approximation. The result is $\mathrm{v(t)} = \frac{\mathrm{m}}{\mathrm{C_d}\frac{\mathrm{p}}{2}\mathrm{A}}\left(\mathrm{t} + \frac{m}{C_d \frac{p}{2} A \mathrm{v_i}}\right)^{-1}$. By taking the indefinite integral of this function, a function that describes the position of the projectile can be given as well. This is $\mathrm{p(t)} = \frac{m}{C_d \frac{p}{2} A} \ln\Big(t +$

$\frac{m}{C_d\frac{p}{2}Av_i}\Big) + \frac{m}{C_d\frac{p}{2}A}\ln\left(\frac{C_d\frac{p}{2}Av_i}{m}\right)$. Both functions are posted on the Desmos link given above, and they each respond to the changes in the variables as expected.

This kind of function models horizontal motion with quadratic drag. This means that the drag force increases as a square of the velocity. Quadratic drag is used in instances where the Reynold's number high. More details can be found at http://www.physics.udel.edu/~szalewic/teach/419/cm08ln_quad-drag.pdf

In the future, I would like to expand these functions to include various angles of launch in a gravitational field, so that a realistic position plot in an x-y plane can be created.